THE PRECISION OF A PERIOD

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Abstract

Formulae are given for a least-squares line on an O-C diagram, with calculation of the period and epoch and their mean errors.

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1. Introduction

At the Cornell meeting of the AAVSO in the spring of 1988, one of the members asked me about the precision of variable-star periods. At the Maria Mitchell Observatory we almost always give a newly revised period in the form "period ± mean error." He showed me the way he had assigned a plus-or-minus value to a period he had calculated. It seemed to give a good estimate of the uncertainty, but it was not as rigorous as using the method of least squares to put a line on an O-C diagram. An optional step in that method gives the mean errors. He suggested that I write up the procedure for this Journal in the form of formulae that would be handy for variable star observers.

Typical presentations of the method either omit the calculation of errors (e.g., Swartz 1973) or they give considerably more information than is needed by a user. (Chauvanet 1893; Brandt 1976). They include proofs, which necessarily rely on calculus, and the formulae tend to be in a notation that looks intimidating, with numerous subscripts or superscripts, and special ways of indicating operations like forming sums of products.

Using the method, as opposed to proving it, is fairly straightforward, although somewhat tedious. This paper is an attempt to present the formulae in the form of a recipe that can be used by anyone who is interested in results and willing to do without the mathematical derivations. No knowledge of calculus is needed to use the recipe. A little algebra is helpful but not really anything beyond recognizing that A=V/W means that the value of A is the result of dividing the value of V by the value of W. Multiplications will be shown in the recipe with an asterisk, as is usual in computer languages. Thus A times X is shown in this paper as A*X, rather than in the usual algebraic form, AX. All of the steps of the recipe are in the numbered equations. The only unknown in each equation at the time it is reached is on the left hand side. The intervening text provides some amplification but is not strictly necessary for using the method.

The procedure described is specifically for least-squares lines in O-C analysis, but the central part of the process, equations (10) through (24), applies to other plots of Y against X as well.

2. Astronomical Background

O-C analysis starts with knowing the approximate period and epoch of a variable star. (The epoch is a chosen time of maximum for a pulsating star, minimum for an eclipser. The text will say "maximum," letting the reader make the substitution if necessary.) The aim is to use observed times of maximum to find 1) corrections to the old period and epoch and 2) the mean errors of the new period and epoch. Information about both is contained in the deviations, O-C, of observed times of maximum from times computed using the old epoch and period.

As each step is introduced it will be applied to observed maxima of V1281 Aql (Nr. 3668 in Gessner 1973) which is classified as a possible W Virginis star (Kholopov 1985).

The first two steps in the recipe are to assemble the elements.

$$M = \text{(old) epoch of maximum} = 7588 \quad (1)$$

$$P = \text{(old) period} = 32.05 \text{ days} \quad (2)$$

The next two are to tabulate the observations.

$$N = \text{number of observed times of maximum} = 9 \quad (3)$$

$$O = \text{observed times of maxima} \quad (4)$$

The nine values of O for our sample star are in Table I, column 1. These and the epoch, M, are in the form JD-2430000. The next step is tricky:

$$E = \text{number of whole cycles between M and O.} \quad (5)$$

For our sample star these, too, are copied directly from Gessner's paper to Table I, column 2. But what if someone had not already assigned these numbers? E is usually the closest integer to the quotient (O-M)/P. Sometimes we are unsure of the number of cycles between widely spaced observations. That happens when we have lost the cycle count through not knowing the period accurately enough. In that case, however, we are not ready for a least-squares refinement.

$$C = \text{computed times of maximum} = M + P*E \quad (6)$$

$$Y = O - C \quad (7)$$

The nine values of C and Y for the sample star are in columns 3 and 5.

The O-C values, called Y here, are well known indications of the quality of the original epoch and period. If a graph of Y against E looks like a line, then the period has been reasonably constant during the time interval covered by the observations, and the characteristics of the line tell us how to improve M and P. If not, then we must abandon the hope of describing the variation with a constant period.

For our sample star, a line looks reasonable. Figure 1 is a plot of Y, not against E, but, for reasons that will become clear later, against a quantity, X, defined by

$$D = \text{average of the N values of E} = 3.556 \quad (8)$$

$$X = E - D. \quad (9)$$

If the observations and the adopted epoch and period had been perfect, then all of the values of Y would be zero and the points would all lie exactly on the X axis. In the real case of V1281 Aql, the points suggest a horizontal line on or near the X axis, but scatter rather widely around it. The fact that the slope of the implied line is near zero means that Gessner's period is a good one. The scatter around the implied line means that the values of Y have uncertainties of perhaps a few days.

Now it is time to leave the astronomy and see how to find the equation of the best line that can be drawn through scattered points. Later we shall return to astronomy to find both the corrections that need to be applied to the old elements, and the mean errors of the new period and epoch.

3. **Least-Squares Lines**

If the points on a graph of Y against X all lie on a line, then the values of Y are related to the corresponding values of X by an equation of the form Y = A*X + B. It would be possible to take any two values of Y and the corresponding values of X, form two equations, and solve for the two unknowns (A and B) by methods taught in algebra. The case under consideration, however, consists of more than two points. The points suggest a line but do not fall exactly on it. No matter how we choose A and B, we cannot make A*X+B come out exactly equal to the corresponding value of Y. We will instead get some nearby value which we shall call L. It defines a point exactly on a line, above or below the observed point.

The method of least squares is designed to choose A and B to make the values of L come out as close to the values of Y as possible. The method is designed specifically for cases, like the one under consideration, where Y is subject to error, while X is known exactly or almost so. How should we choose A and B? Various authors (e.g., Chauvanet 1893, sections 28 and 29) prove that the best values of A and B in this case are the ones which satisfy these two equations:

W*A + Z*B = V

Z*A + N*B = U,

in which U, V, W, and Z have the values given by the next steps of the recipe.

U = sum of the N values of Y = 0.40 (10)

V = sum of the N values of X*Y = 86.2 (11)

W = sum of the N values of X^2 = 47994 (12)

Z = sum of the N values of X = 0.00 (13)

The sum of the 9 values of X did not come out to be exactly zero in Table I, but that is only because the values of X had to be rounded. Z is really exactly zero because of the way X is defined: E minus the average value of E. One of the reasons for using X instead of E in the least squares solution is that the equations for A and B become simpler when Z is zero:

W*A = V

N*B = U.

Now each equation has only one unknown. The recipe steps for A and B are straightforward:

A = V/W = 0.0018 (14)
B = U/N = 0.044 (15)

The question of rounding has come up. How do we know how many places to carry in the numerical work? Like most workers, I often carry more places than needed. In fact it is easiest with a calculator or computer not to do any rounding of intermediate results if the machine can use them without the operator having to write them down. For the purposes of Table I and the numerical values given in the steps of the recipe, I printed out all intermediate values, rounding as seemed appropriate, usually carrying one or two more places than are significant. It is always good to carry at least one extra, to make sure that rounding errors remain small compared with the observational errors. We shall know at the end which ones were extra, when we see the size of the real errors.

In order to find the errors, we need first to see how close the line comes to the points. Points on the line all satisfy the equation

$$L = A*X + B. \tag{16}$$

For each value of X we get a value of L (column 8) which ought to be close to the corresponding value of Y. Their difference is called the residual of the point from the line:

$$R = Y - L. \tag{17}$$

The sum of the squares of these residuals (column 9 and 10) is an important quantity. When A and B are chosen by the formulae of least squares, then this sum is as small as possible, a circumstance which has given the method its name. The next steps in the recipe are to find and use that sum.

$$Q = \text{sum of the N values of } R^2 = 48.80 \tag{18}$$

$$F = Q/(N-2) = 6.971 \tag{19}$$

$$G = F/W = 0.000145 \tag{20}$$

$$H = F/N = 0.7746 \tag{21}$$

The mean errors are

$$\text{m.e. of } Y = \text{square root of } F = 2.6 \tag{22}$$

$$\text{m.e. of } A = \text{square root of } G = 0.012 \tag{23}$$

$$\text{m.e. of } B = \text{square root of } H = 0.88. \tag{24}$$

At this point we know the equation of the line and the precision with which it is known. Equation (16), with the values of A and B from equations (14) and (15), was used to plot the line in Figure 1.

Equations (10) through (24) may be used for any least-squares line, not just an O-C diagram, provided that the errors are in Y but not in X, and provided further that the value of Z has been set to zero as this recipe did in equations (8) and (9). A few more steps are needed to produce the astronomical results which were the motivation for the calculation.

4. **The New Elements and Their Errors**

It can be shown that the new linear elements of the variable star are

$$\text{New period} = \text{Old period} + A \tag{25}$$

$$\text{New epoch} = \text{Old epoch} + B - A*D. \tag{26}$$

The mean error of the new period is

$$\text{m.e. of the new period} = \text{m.e. of } A = 0.012. \tag{27}$$

The mean error of the new epoch is rather more complicated since the epoch depends on both A and B unless D happens to be zero. First let

$$S = H + G*D2 = 0.7764 \tag{28}$$

then,

m.e. of the new epoch = square root of S = 0.88. (29)

Equations (28) and (29) do not apply to all least squares lines but only when the sum of the values of X is zero, another good reason for defining X by equation (9). In the case of V1281 Aql, the new elements and their mean errors are

New Period = 32.052 $\pm$ 0.012

New Epoch = 7588.02 $\pm$ 0.88.

It would be all right to round the period to 2 decimal places, but it is considered best to quote enough digits so that the last one is uncertain by more than just one or two units. Considering the size of the mean error of the epoch, however, we are not justified in carrying 2 decimal places the new epoch. The astronomical result of all this calculation is, then,

$$JD_{(max)} = \underset{\pm 0.9}{2437588.0} + \underset{\pm 0.012}{32.052}\, E.$$

The only real improvement over Gessner's result, which was, of course, based on the same data, is that the method of least squares has given us the precision of the elements.

The notation seems to imply that the new period and epoch are required to lie in the indicated ranges, but no such implication is intended. The theory of least squares and the definition of mean error state that the true (and unknowable) values of the quantities have about a 68% probability of lying in the range defined by their mean errors. The mean error of Y is the measure of how close the points do come to the line. If the residuals are due to observational errors then 68% of a large number of observed points should fall no farther from the line, vertically, than $\pm$ the mean error of Y.

A few words about the source of these deviations. The mean error of Y is calculated from the residuals, R. It evaluates the disagreement between the line and the observations, without placing the blame on either the line or the observations. It is sometimes called observational error, and the 68% probability in the previous paragraph refers to the case of random errors such as inevitably afflict observations. But what if the period is not constant? Then even the best line provides only some sort of an an average period, and the residuals contain a contribution from real change in period. In the case of V1281 Aql the mean error came out to be 2.6 days. While this is nearly 10% of the period, it is not surprisingly large for this star. Not only does the light curve have a rather broad maximum, which prevents O from being sharply defined, but Gessner also suggests that the type may be semiregular (SRd) rather than W Virginis. Individual cycles may well have been longer or shorter than 32.052 days.

In our sample case, the deviations from the line look random. In some cases they show a systematic trend (e.g., Provencal 1986). The next most complicated least-squares situation is to put a parabola through the points on an O-C diagram, indicative of a constant rate of change. That means calculating three quantities. Instead of equations (14) and (15) for A and B, there will be three equations for three unknowns and, no matter how we define X, there will be at least two unknowns in each equation. Having already almost exhausted the alphabet, this recipe ends with the line.

An instructive final application of linear least squares is to see what would have happened if the first observation, at E = -194, had not been available. The result of a recalculation of equations (3) through (29) with the remaining 8 observed maxima is

$$JD_{(max)} = 2437588.0 + 32.052\ E.$$
$$\pm 1.6 \quad \pm 0.045$$

The much larger mean error of the period is an example of a situation familiar to variable star workers. The period of a star (specifically the average period over a number of cycles) is more precisely known when the number of cycles is greater. The larger mean error of the epoch is more apparent than real. An epoch is most reliably determined when it has been chosen to lie near the average value of the observed times of maximum, rather than significantly earlier, as in this case, or significantly later. An option for variable star workers is to add (or subtract) an appropriate number of cycles to (or from) the original epoch, then use least squares to improve the resulting alternative epoch.

5. Summary

A method has been given, without derivation, for the improvement of linear elements through least squares analysis of O-C data, with emphasis on the determination of the mean errors of the revised elements. The steps are in the numbered equations (1) through (29) in the order in which they are to be carried out. The central part of the procedure, equations (10) through (24), is applicable to least-squares lines in general. Computer programs using this method are in use at the Maria Mitchell Observatory in our studies of periodic variable stars. Our variable star research receives support from National Science Foundation grant AST86-19885.

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TABLE I

V1281 Aql: Least-Squares Calculation of Linear Elements

O	E	C	X	Y	X^2	X*Y	L	R	R^2
1370	-194	1370.30	-197.556	-0.30	39028.4	59.27	-0.31	0.01	0.00
7588	0	7588.00	-3.556	0.00	12.6	0.00	0.04	-0.04	0.00
7875	9	7876.45	5.444	-1.45	29.6	-7.89	0.05	-1.50	2.25
7910	10	7908.50	6.444	1.50	41.5	9.67	0.06	1.44	2.07
7940	11	7940.55	7.444	-0.55	55.4	-4.09	0.06	-0.61	0.37
8675	34	8677.70	30.444	-2.70	926.8	-82.20	0.10	-2.80	7.84
8940	42	8934.10	38.444	5.90	1477.9	226.82	0.11	5.79	33.52
9350	55	9350.75	51.444	-0.75	2646.5	-38.58	0.14	-0.89	0.79
9670	65	9671.25	61.444	-1.25	3775.4	-76.81	0.15	-1.40	1.96
SUMS	32		-0.004	0.40	47994.1	86.19		0.00	48.80

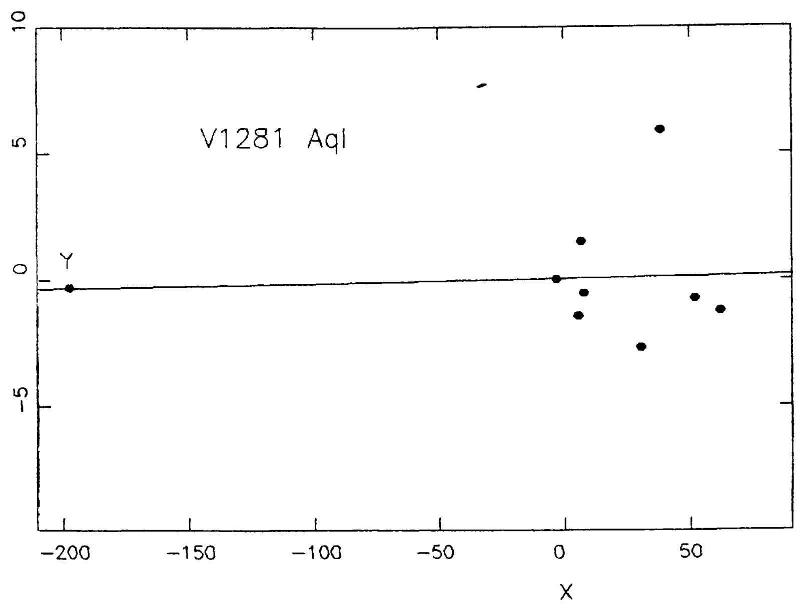

Figure 1. O-C diagram for V1281 Aql. Y is O-C; X is related to the cycle count, E, through equations (8) and (9). The least-squares line is shown.