Methods for O–C (Observed Minus Computed) Diagrams and for the Determination of Light Elements of Variable Stars with Linear and Second Order Polynomial Ephemerides

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Abstract   Methods are described for the construction and analysis of O–C (observed minus computed) diagrams and for the determination of light elements (ephemerides) of variable stars and the standard errors of the elements. The methods described are those that apply: (1) when the period of the star is constant, and (2) when the period of the star is changing continuously, and the light elements can be represented by a second order polynomial function.

1. Introduction

During the past few years, the author performed photometric studies on the monoperiodic high amplitude δ Scuti stars RS Gruis and BS Aquarii (Axelsen 2014a, 2014b). Both of these stars have been studied by several observers over of period of decades, and both exhibit evidence of changes in period (Rodríguez *et al.* 1995; García 2012; Axelsen 2014a; Yang *et al.* 1993; Fu *et al.* 1997; Axelsen 2014b). For RS Gru, historical data and data in recent years revealed period changes that are best expressed as quadratic (second order polynomial) functions, indicating that there have been period changes, but the rates of period change are constant (Rodríguez *et al.* 1995; Axelsen 2014b). The recent behavior of BS Aqr, on the other hand, is best expressed by a linear function, indicating no continuous change in period over recent decades, although period change had occurred previously (Axelsen 2014a).

The author found no entirely satisfactory references in the literature which provided detailed, step by step methods for these types of analysis. Online lecture notes by Bradstreet (1997) on the ephemerides of variable stars did, however, provide some assistance. The steps involved are: tabulation of times of maximum light (TsOM, the times of peak brightness in the light curve) of the variable star in heliocentric Julian days (HJD); calculation of the O–C values; drawing and analysis of the O–C diagram; determination of the light elements for the variable star, and the associated standard errors; and (if the period of a star changes continuously over time) calculation of the rate of change of the period, using appropriate units in which to express the rate of change.

This paper therefore presents the author's personal approach to the analysis of longitudinal data comprising the times of maximum of variable stars with regular periods. The data used as examples are available in the astronomical

literature. Calculations performed on those data are described in this paper, and the results compared with those previously published. The calculations can be replicated by using the methods described herein.

In principle, the calculations can also be applied to times of minimum of eclipsing binary systems. For simplicity of description, however, only times of maximum, such as can be determined for many pulsating variable stars, will be referred to.

2. Data

In the case of BS Aqr, data from the astronomical literature including recent personal observations are used, with all sources referenced in Axelsen (2014a). In the case of RS Gru, data from the astronomical literature are used, with all sources referenced in Rodríguez *et al.* 1995.

3. Analysis, O–C diagrams, and light elements

3.1. Linear ephemeris

The data and the results of calculations of O–C values are listed in Table 1 for BS Aqr, which represent data already published as maxima 35 to 61 of Table 2 in Axelsen (2014a).

The requirements for the calculation of O–C values are the availability of several or many TsOM of a variable star, which may comprise historical data previously reported in the astronomical literature, and/or recent observations. TsOM are in heliocentric Julian days.

The parameters used in the calculation are:

T_0 The initial TOM in heliocentric Julian days (HJD), obtained either recently or at some time in the past. This is the TOM on which the calculations are based.

P_{est} The estimated period in days, obtained either recently or at some time in the past. This is the period on which the calculations are based.

T_n Other TsOM in HJD, from historical data and/or recent observations.

E_n The number of cycles or epochs of the variable star that have elapsed between T_0 and each value of T_n.

It should be explained what each O–C value represents. The "O" (the observed values) represent the observed TsOM, either from the literature or personal observation. If there are sufficient observations of the magnitude of the star on either side of each peak in the light curve, an excellent way to determine each TOM is to fit a 6th-order polynomial expression to the ascending limb, the peak, and the descending limb of the curve. The software program PERANSO (Vanmunster 2013) has a routine for this determination, and provides also the

Table 1. Times of maximum (TsOM) in heliocentric Julian days (HJD) of BS Aqr from 1973 to 2013, representing maxima 35 to 61 from Table 2 of Axelsen (2014a).

Max	TOM (HJD) T_n	Epoch E (Rounded)	Epoch E_n	(O–C)	Primary Source*
35	2441946.67140	−73785	−73785.06973	−0.01379	1
36	2441946.86930	−73784	−73784.06934	−0.01372	1
37	2441947.66200	−73780	−73780.06221	−0.01231	1
38	2441947.86030	−73779	−73779.05980	−0.01183	1
39	2441948.65000	−73775	−73775.06784	−0.01342	1
40	2441948.84890	−73774	−73774.06239	−0.01234	1
41	2441949.64000	−73770	−73770.06335	−0.01253	1
42	2441950.63000	−73765	−73765.05887	−0.01165	1
43	2441950.82950	−73764	−73764.05039	−0.00997	1
44	2445612.72400	−55253	−55253.04974	−0.00984	2
45	2445620.63800	−55213	−55213.04420	−0.00874	2
46	2445625.58300	−55188	−55188.04706	−0.00931	2
47	2445637.64700	−55127	−55127.06313	−0.01249	2
48	2445644.57200	−55092	−55092.05702	−0.01128	2
49	2445997.09200	−53310	−53310.05649	−0.01117	3
50	2448912.80460	−38571	−38571.03060	−0.00605	5
51	2449606.76440	−35063	−35063.04032	−0.00798	5
52	2449606.96210	−35062	−35062.04094	−0.00810	5
53	2449607.95130	−35057	−35057.04050	−0.00801	5
54	2449956.51500	−33295	−33295.03919	−0.00775	5
55	2449957.50480	−33290	−33290.03572	−0.00707	5
56	2450072.04410	−32711	−32711.03568	−0.00706	4
57	2456543.02662	0	0.00000	0.00000	6
58	2456545.00339	10	9.99264	−0.00146	6
59	2456561.02607	91	90.98783	−0.00241	6
60	2456570.12942	137	137.00557	0.00110	6
61	2456592.08450	248	247.98924	−0.00213	6

Primary sources: 1. Elst (1976); 2. Meylan et al. *(1986); 3. Yang* et al. *(1993); 4. Fu* et al. *(1997); 5. Rodríguez* et al. *(1998); 6. Axelsen (2014a).*

Notes: The epochs (number of cycles) are listed with the exact calculated results in the fourth column, and the rounded numbers (from which the O–C values are calculated) in the third column. The last two columns list the O–C (observed minus computed) values and the published sources of the TsOM, respectively. The calculations represent the results of formulae (1) and (2) in the present paper, with P_{est} being the period of Fu et al. *(1997) of 0.197822612 d, and T_0 being the first TOM determined by us in 2013 (maximum 57 in the table), 2456543.026620 HJD.*

standard error of the calculated result. The "C" (the computed values) represent the TsOM of the light curve that would have occurred if P_{est}, the initial estimated period, remained unchanged during the span of time between T_0 and T_n (the

span of time during which the star has been studied). If the period is, in fact, unchanged, then the O–C value will be zero. On the other hand, if the period has changed, O–C will be a non-zero value.

The first calculation which must be made determines the number of cyles or epochs (E_n) of the star which have elapsed between T_0 and T_n, and is represented by the formula:

$$E_n = (T_n - T_0) / P_{est} \tag{1}$$

The resulting numbers, E_n, will be numerals with decimal fractions, as seen in the fourth column of Table 1, and may be positive or negative (or both), depending on which TOM is designated as T_0. These numbers must be reduced to integers, which can be achieved by rounding, shown as the Epoch E in the third column of Table 1.

After the epochs (number of cycles) E in integer format are tabulated, the O–C values can be calculated from the formula:

$$O–C = T_n - (T_0 + E \times P_{est}) \tag{2}$$

All of the above calculations can be made readily using a spreadsheet, and from that spreadsheet the O–C diagram can be charted, with E (epochs) on the x-axis and the O–C values on the y axis.

Figure 1 is an O–C diagram drawn from the data in Table 1. The line represents a least squares linear fit, calculated using Data Analysis, Regression in MICROSOFT EXCEL, and is represented by the equation (with standard errors in brackets):

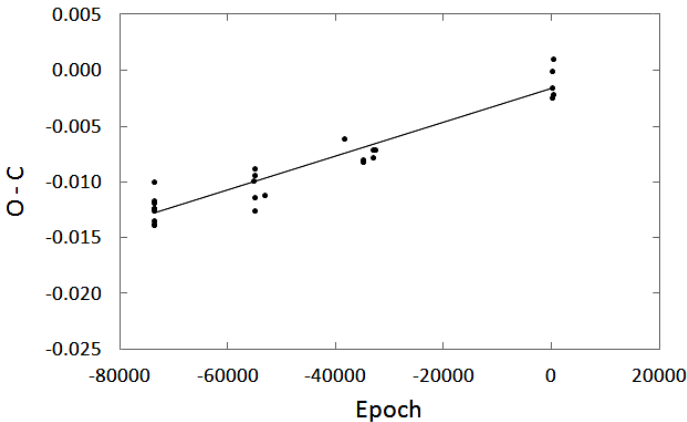$$O–C = 0.00000015 \ (1) \ E - 0.0016 \ (5) \tag{3}$$

Figure 1. O–C diagram of BS Aqr, drawn from the data of Table 1. The line represents a least squares linear function fitted to the data.

Since the O–C diagram in this example is described by a linear function, the light elements of the star are represented by an equation that describes a linear function. This function can be derived by plotting each TOM on the y-axis against each epoch E on the x-axis (using the data from Table 1), and finding the least squares linear regression of this plot. This type of plot is not usually presented in the literature, but its equation represents the light elements of the star:

$$\text{TOM (HJD)} = 0.19782276 \ (5) \ E + 2456543.0250 \ (5) \tag{4}$$

The slope of the function is the period, seen above as 0.19782276 d. The last term of the equation (2456543.0250) is the TOM at zero epoch. Standard errors are shown in parentheses. The TOM at any user-nominated epoch can be calculated directly from the above equation by inserting the required value of E.

Since the function fitted to the O–C data is linear, the period of the star has not been changing continuously during the period of the analysis. Since the fitted function has a positive slope, the calculated period of the star is longer than the estimated period P_{est} used to make the calculations of O–C and the number of epochs. A negative slope would indicate that the calculated period is less than the estimated period P_{est}, and a horizontal line would indicate that the calculated period is the same as the estimated period P_{est}.

3.2. Ephemeris described by a second-order polynomial function

The data and the results of calculations of O–C values are listed in Table 2 for RS Gru. The table uses the TsOM and epochs published by Rodríguez *et al.* (1995), and also lists the O–C values calculated by the present author from the initial period and epoch used by Rodríguez *et al.* (1995), namely, 0.14701131 d and HJD 2447464.7095. The O–C diagram from this data is shown in Figure 2, together with the least squares fit from a second-order polynomial function, represented by the generic equation:

$$y = ax^2 + bx + c \tag{5}$$

and specifically by the equation:

$$\text{O–C} = -3.14 \times 10^{-12} \ E^2 - 4.46 \times 10^{-7} \ E + 5.92 \times 10^{-4} \tag{6}$$

where O–C represents the values on the y-axis, and E represents the epochs, plotted on the x-axis.

It should be noted that, in the paper by Rodríguez *et al.* (1995), the values listed in the columns under the headings for O–C in their Table 2 are not actually the O–C values themselves, but the residuals from the linear and quadratic regression analysis performed by those authors. Likewise, their O–C diagram does not plot the values themselves, but the residuals of the O–C values from the second-order polynomial (quadratic) least squares fit.

Table 2. Times of maximum (TsOM) of RS Gru from 1952 to 1988, epochs (number of cycles, with only the rounded values shown), and O–C values.

Max	TOM (HJD) T_n	Epoch E (Rounded)	(O–C)	Primary Source*
1	2434325.2940	–89377	–0.026108	1
2	2434573.4510	–87689	–0.023447	1
3	2436756.5710	–72839	–0.014777	2
4	2436760.5380	–72812	–0.017070	2
5	2436801.5540	–72533	–0.017101	3
6	2436853.3030	–72181	–0.015926	3
7	2441538.4027	–40312	–0.005450	4
8	2441538.5490	–40311	–0.006161	4
9	2441610.4379	–39822	–0.005574	4
10	2441611.3200	–39816	–0.005539	4
11	2441611.4677	–39815	–0.004850	4
12	2441612.3493	–39809	–0.005315	4
13	2441915.4856	–37747	–0.005417	4
14	2442687.5892	–32495	–0.002874	5
15	2443355.4610	–27952	–0.001429	6
16	2443355.6092	–27951	–0.000240	6
17	2443360.4584	–27918	–0.002399	6
18	2443360.6050	–27917	–0.002810	6
19	2447464.7095	0	–0.000600	7
20	2447468.5324	26	0.000018	7
21	2447468.6793	27	–0.000093	7
22	2447472.6489	54	0.000213	7

*1. Hoffmeister (1956); 2. Oosterhoff and Walraven (1966); 3. Kinman (1961); 4. Dean et al. (1977); 5. McNamara and Feltz (1976); 6. Balona and Martin (1978); 7. Rodríguez et al. (1995).

Notes: The times of maximum are those published by Rodríguez et al. (1995). The calculations, employing formulae (1) and (2) of the present paper, use the initial TOM and period published by the same authors, namely, 2447464.7101 HJD, and 0.147010864 d, respectively.

The importance of the O–C diagram both in Rodríguez *et al.* (1995) and the present paper is that it shows a second-order polynomial function fitted to the data, with the plot of that function being a curve with its concavity facing down. This implies that the period of RS Gru had been continuously decreasing during the span of time covered by the data, from 1952 to 1988. If the plot were a curve with the concavity facing up, the period would have been continuously increasing.

Therefore, the light elements of the star should similarly be described by a second-order polynomial expression, obtained from a graph of the TsOM plotted on the y-axis against epochs E on the x-axis, with the values taken from
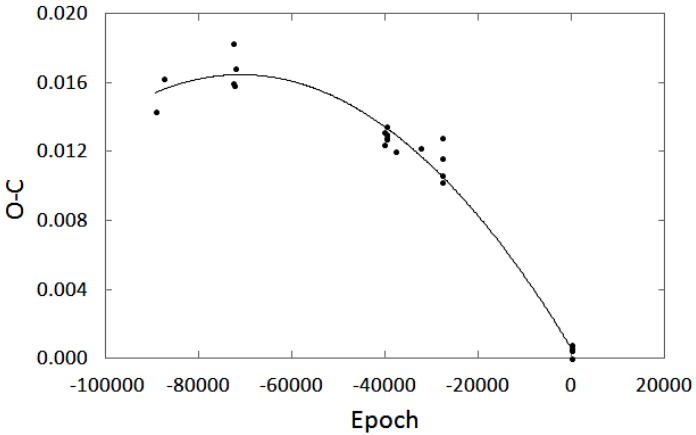
Figure 2. O–C diagram of RS Gru, drawn from the TsOM and epochs published by Rodríguez *et al.* (1995), and listed in Table 2 of the present paper. The curved line represents the second-order polynomial expression fitted to the data.

Table 2. When the data are plotted in this manner, the relationship appears to be linear and is accurately described by a linear function, but we know from the O–C diagram that it should be described by a second-order polynomial expression in order to obtain accurate light elements.

The second-order polynomial expression calculated by the present author for these data using Data Analysis, Regression in MICROSOFT EXCEL is:

$$TOM = -3.14 \times 10^{-12} \ (\pm 0.24 \times 10^{-12}) \ E^2$$
$$+ \ 0.147010864 \ (22) \ E + 2447464.7101 \ (4) \tag{7}$$

This expression involves a term that is the square of the epoch. In order to use MICROSOFT EXCEL to perform a least squares second-order polynomial regression, both the epoch and the square of the epoch must first be tabulated so that they can be selected as input for the procedure in EXCEL. Table 3 lists these data, from which Data Analysis, Regression calculated the coefficients shown in Equation 7 above. These coefficients are identical to those published by Rodríguez *et al.* (1995) in their analysis of the same data. This expression represents the light elements of RS Gru, as explained below.

Equation 7 has the generic form $y = ax^2 + bx + c$, already mentioned above as Equation 5. When this generic equation represents the light elements of a variable star, it is best understood when written as:

$$TOM = AE^2 + PE + T_1 \tag{8}$$

Here, TOM is the calculated time of maximum in HJD for any input epoch E. P is the period of the star in days at zero epoch, and T_1 is the TOM at zero epoch. If both the period and the TOM are required for any epoch, they can be calculated

Table 3. RS Gru data, with epochs and times of maximum in the second and third columns (values taken from Table 2 above), and the values of the epoch squared in the first column.

Epoch Squared	Epoch	TOM (HJD)
7988248129	−89377	2434325.2940
7689360721	−87689	2434573.4510
5305519921	−72839	2436756.5710
5301587344	−72812	2436760.5380
5261036089	−72533	2436801.5540
5210096761	−72181	2436853.3030
1625057344	−40312	2441538.4027
1624976721	−40311	2441538.5490
1585791684	−39822	2441610.4379
1585313856	−39816	2441611.3200
1585234225	−39815	2441611.4677
1584756481	−39809	2441612.3493
1424836009	−37747	2441915.4856
1055925025	−32495	2442687.5892
781314304	−27952	2443355.4610
781258401	−27951	2443355.6092
779414724	−27918	2443360.4584
779358889	−27917	2443360.6050
0	0	2447464.7095
676	26	2447468.5324
729	27	2447468.6793
2916	54	2447472.6489

Notes: This table represents the inputs for regression analysis of a second-order polynomial expression. Using "Data Analysis, Regression" in MICROSOFT EXCEL, the input y range is the third column, the TsOM. The first two columns, selected together, represent the input x range.

by rescaling the x-axis values so that the epoch of interest is assigned the value zero. Importantly, the coefficient A is related to the rate of change of the period of the star, with the actual rate of change represented by the first differential of this coefficient, which is 2A. This value is not changed if the x-axis values are rescaled to change the zero epoch. It should be noted that the coefficient of the E^2 term in the equation for the ephemeris is the same as the coefficient of the E^2 term in the equation for the O–C diagram (Equation 6 above).

Finally, the results for the rate of change of the period of the star need to be expressed in the appropriate units, as usually reported in the literature. The coefficient $−3.14 \times 10^{-12}$ in Equation 7 above has the units d/cycle (days per cycle), since the TsOM on the y-axis are expressed in HJD, and the number of cycles (or epochs) are on the x-axis. Table 4 lists the sequence of arithmetical steps to show, first, how the coefficient is multiplied by 2 to yield the actual

Table 4. Calculations made by the present author which yield the appropriate units in which to express the rate of change of the period of RS Gru for the years 1952–1988.

Term	Coefficient	SE	Units
A	-3.138×10^{-12}	2.428×10^{-13}	d/cycle
2A	-6.275×10^{-12}	4.857×10^{-13}	d/cycle
Divide by period	-4.269×10^{-11}	3.304×10^{-12}	d/d
Multiply by 365.25	-1.56×10^{-8}	0.12×10^{-8}	d/yr or d yr^{-1}
Divide by period	-10.6×10^{-8}	0.8×10^{-8}	cycle/yr or yr^{-1}

Notes: A = the term in equation (8) TOM = AE² + PE + T₁ which relates to the rate of change of the period of the star; SE = standard error; d = day; yr = year. The results in the fourth and fifth lines of the table represent the units usually reported in the literature, designated as dP/dT and dP/PdT, respectively. The values above were calculated by the author. The results in the fourth and fifth lines of the table are identical to those published by Rodríguez et al. *(1995).*

rate of change of the period, and then how that value (in units of d/cycle) is progressively converted into dP/dT days per year (d/yr, or d yr^{-1}) and finally dP/PdT cycles per year (cycles/yr or simply y^{-1}). The final values in the last two lines of Table 4, calculated by the present author, are identical to those reported by Rodíguez *et al.* (1995).

4. Conclusions

The aim of this paper is to provide a step by step guide to the analysis of data comprising the times of maximum of variable stars over a substantial period of time, usually years, and which can be examined by the construction of O–C diagrams. A linear O–C plot indicates that the period of the star is not changing during the span of time covered by the analysis. A linear function with a positive slope indicates that the newly determined period of the star is longer than the original estimate, and a linear function with a negative slope indicates that the newly determined period is shorter than that originally estimated. Functions that represent the light elements (the ephemeris) of the star are shown as plots of TsOM on the y axis against epoch (cycle number) on the x-axis, although the actual plot is usually not shown in publications. The slope of the linear function represents the period of the star in days. Where the period of a star is changing continuously, and the O–C diagram can be represented accurately by a second-order polynomial expression, the light elements are also best described by such a polynomial function, which allows an accurate determination of the period at any epoch, the TOM at any epoch, and the rate of change in the period.

5. Acknowledgements

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